

ROBUST TRANSITIVITY AND DENSITY OF PERIODIC POINTS OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

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ABSTRACT. We prove results related to robust transitivity and density of periodic points of Partially Hyperbolic Diffeomorphisms under conditions involving Accessibility and a property in the tangent bundle .

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1. INTRODUCTION

A very interesting feature of a differentiable dynamical system is topological transitivity. Being a sign of complexity of the underlying dynamics it prevents the possibility of reducing its study to simplest systems.

One of the most important questions in the theory of Differentiable Dynamical Systems regarding a particular dynamical property is to recognize when it is present in all nearby systems (with respect to some topology). When this happens we say that the property is robust or stable under perturbations.

So the search for conditions on a differentiable dynamical system leading to robust transitivity has been a topic of interest for a long time. Many examples exhibiting robust transitivity has been studied, beginning with the transitive Anosov diffeomorphisms.

Robust transitivity is not an exclusive property of Hyperbolic Diffeomorphisms as it has been showed first by the example of Shub on the torus \mathbb{T}^4 , later by the example of Mañé on the torus \mathbb{T}^3 and more recently by the example of Bonatti and Días in [?]. All these examples are Partially Hyperbolic Systems (see section 2). While other example, due to Bonatti and Viana [?], exhibits just dominated splitting.

Ergodicity and its stability are other important properties to study on a dynamical system. A well known conjecture formulated by Pugh and Shub [?] on Stable Ergodicity for Partially Hyperbolic Systems has been the motivation for a lot of research during the last few years.

One of the conditions appearing on the hypothesis of this conjecture is that of Accessibility (see section 2) which as the work [?] shows is a typical property in the sense that it is C^1 dense among the C^r Partially Hyperbolic Diffeomorphisms of a compact manifold.

Accessibility has also a relation with transitivity according to Brin's Theorem [?] stating that in a Partially Hyperbolic Accessible System, transitivity is equivalent to the fact of the non-wandering set being the whole manifold.

In connection with the mentioned examples of Shub and Mañé, the authors Pujals and Sambarino introduced in [?] an interesting property which they call Property SH, to guarantee that the strong stable foliation is robustly minimal. A key feature of Property SH is its intrinsic robustness which makes it an appealing condition to use for establishing robust transitivity in more general contexts.

This work has been motivated by the idea of exploring the consequences, in the sense of robust transitivity, of the combination of Property SH and Accessibility, for Partially Hyperbolic Systems.

Our first result arise naturally after the observation that in the proof of Brin's Theorem the accessibility in relation to open sets (as defined in Section 2) is enough to guarantee transitivity. Having at hand the Property SH and its robustness we can establish the robustness of accessibility in relation to open sets, see Corollary 4.1. As a Corollary, it follows our first result (see Section 4) related to robust transitivity:

Corollary 1.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic, accessible, volume preserving diffeomorphism exhibiting the property SH. Then any diffeomorphism C^1 -close to f that is volume preserving is topologically mixing.*

Abdenur and Crovisier proved in [?] that the fact of a diffeomorphism being robustly transitive implies that it is also topologically mixing modulo an arbitrarily small C^1 perturbation. In the same spirit we have the following Theorem in Section 3:

Theorem 1.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism, accessible in relation to open sets, satisfying $\Omega(f) = M$ and the Property SH, then f is topologically mixing.*

In section 7 we show that Shub's example in \mathbb{T}^4 satisfies all the conditions in Theorem 1.1.

Theorem 1.2 by Miss Oliveira proves that Property SH is enough to guarantee robust transitivity.

Theorem 1.2. *Let M be a compact Riemannian manifold and let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism, non-hyperbolic, transitive. If f and f^{-1} satisfy Property SH then f is robustly transitive.*

The point in this Theorem is that the hypotheses are given only on the tangent bundle. The condition of minimality of the stable foliation assumed in [?] was substituted here by the Property SH for f^{-1} . In section 7 we give an scenario where all the conditions in Theorem 1.2 are realized. An interesting question not addressed in our work is if it is possible to exploit Theorem 1.2 to produce new examples of robustly transitive diffeomorphisms.

Finally we would like to express our deep gratitude to professor Enrique Pujals from IMPA for the multiple suggestions and clarifying discussions during the course of our work.

In the following sections M will denote a compact Riemannian manifold and $\text{Diff}^r(M)$ the set of C^r -diffeomorphisms defined on M .

2. PRELIMINARIES

In this section we recall some well-known results regarding partially hyperbolic systems. We refer to [?], [?], [?], [?] for a general background on the topics we will review.

2.1. Partially Hyperbolic Diffeomorphisms.

As Property SH plays a central role in this work we follow closely the definitions and basic results in [?].

Definition 2.1. A diffeomorphism $f : M \rightarrow M$ is partially hyperbolic provided the tangent bundle splits into three non-trivial sub-bundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$ which are invariant under the tangent map Df and there are $0 < \lambda < \mu < 1$ such that for all $x \in M$

$$\|Df|_{E^{ss}(x)}\| < \lambda, \quad \|Df|_{E^{uu}(x)}^{-1}\| < \lambda, \quad \mu < \|Df|_{E^c(x)}^{-1}\|, \quad \mu < \|Df|_{E^c(x)}\|.$$

Lemma 2.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism and $Gr_k(M)$ denote the Grassmannian bundle of M of k -dimensional spaces. Then there exist a C^r neighbourhood of f , say \mathcal{U} , numbers λ_1 and μ_1 with $0 < \lambda < \lambda_1 < \mu < \mu_1 < 1$ and continuous functions $E^{ss} : \mathcal{U} \rightarrow C(M, Gr_{\dim(E^{ss})}(M))$, $E^c : \mathcal{U} \rightarrow C(M, Gr_{\dim(E^c)}(M))$ and $E^{uu} : \mathcal{U} \rightarrow C(M, Gr_{\dim(E^{uu})}(M))$ such that, for any $g \in \mathcal{U}$ and $x \in M$, we have the following:*

(1) $TM = E^{ss}(g)(M) \oplus E^c(g)(M) \oplus E^{uu}(g)(M)$, this decomposition is invariant under Dg and no one of these sub-bundles is trivial

(2) $\|Dg|_{E^{ss}(x,g)}\| < \lambda_1$, $\|Dg|_{E^{uu}(x,g)}^{-1}\| < \lambda_1$

(3) $\mu_1 < \|Dg|_{E^c(x,g)}^{-1}\|$, $\mu_1 < \|Dg|_{E^c(x,g)}\|$

The sub-bundles $E^{ss}(g)(M)$ and $E^{uu}(g)(M)$ are uniquely integrable and form two foliations \mathcal{F}^{ss} and \mathcal{F}^{uu} .

Theorem 2.1. *Let \mathcal{U} be as in Lemma 2.1. Then, for each $g \in \mathcal{U}$ there are two partitions $\mathcal{F}^{ss}(g)$ and $\mathcal{F}^{uu}(g)$ of M such that for each $x \in M$ the elements of the partitions that contain x , denoted by $\mathcal{F}^{ss}(x, g)$ and $\mathcal{F}^{uu}(x, g)$ are C^1 submanifolds such that $T_x \mathcal{F}^{ss}(x, g) = E^{ss}(x, g)$ and $T_x \mathcal{F}^{uu}(x, g) = E^{uu}(x, g)$. These submanifolds depend continuously (on compact subsets) on $x \in M$ and $g \in \mathcal{U}$.*

These submanifolds $\mathcal{F}^{ss}(x, g)$ and $\mathcal{F}^{uu}(x, g)$ inherit the Riemannian metric on M . We shall denote by $\mathcal{F}_r^{ss}(x, g)$ (respectively $\mathcal{F}_r^{uu}(x, g)$) the ball in $\mathcal{F}^{ss}(x, g)$ (respectively $\mathcal{F}^{uu}(x, g)$) of radius r centred at x .

The sub-bundle $E^{cu} = E^c \oplus E^{uu}$ is not integrable in general. However, we can choose a continuous family of locally invariant manifolds tangent to it. Let $\dim E^{cu} = l$ and denote by I_ϵ the ball of radius ϵ in R^l .

Lemma 2.2. *Let \mathcal{U} be as in Lemma 2.1 and $\text{Emb}_1(I_1, M)$ the set of C^1 -embeddings of I_1 in M . There exists a continuous map $\varphi : M \times \mathcal{U} \rightarrow \text{Emb}_1(I_1, M)$ such that, if we set $W_\epsilon^{cu}(x, g) = \varphi(x, g)I_\epsilon$, then the following hold:*

(1) $T_x W_\epsilon^{cu}(x, g) = E^{cu}(x, g)$;

(2) given $\epsilon > 0$ there exists $r = r(\epsilon) > 0$ such that $g^{-1}(W_r^{cu}(x, g)) \subset W_\epsilon^{cu}(g^{-1}(x), g)$.

For the sake of simplicity we shall identify $W_\epsilon^{cu}(x, g)$ with the ball of radius ϵ in $W_1^{cu}(x, g)$.

Lemma 2.3. *Let \mathcal{U} be as in Lemmas 2.1 and 2.2. Given $0 < \lambda < \lambda_1 < 1$ there exists r_0 such that if $g \in \mathcal{U}$ and $x \in M$ satisfy*

$$\prod_{j=0}^n \|Dg|_{E^{cu}(g^{-j}(x))}^{-1}\| < \lambda^n, \quad 0 \leq n \leq m,$$

Then $g^{-m}(W_{r_0}^{cu}(x, g)) \subset W_{\lambda_1^m r_0}^{cu}(g^{-m}(x), g)$.

In the following, we will work with partially hyperbolic diffeomorphisms.

2.2. Accessibility.

Definition 2.2. Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. Two points $p, q \in M$ are called accessible, if there are points $z_0 = p, z_1, \dots, z_{l-1}, z_l = q, z_i \in M$, such that $z_i \in \mathcal{F}^\alpha(z_{i-1}, f)$ for $i = 1, \dots, l$ and $\alpha = ss$ or uu .

The collection of points z_0, z_1, \dots, z_l is called the *us-path* connecting p and q . Accessibility is an equivalence relation and the collection of points accessible from a given point p is called the accessibility class of p . We will denote this class by $\mathcal{C}(p, f)$. The diffeomorphism f is said to have the accessibility property if the accessibility class of any point is the whole manifold M , or, in other words, if any two points in M are accessible.

Next we introduce the notion of accessibility in relation to open sets, which we use to give a stronger version of Brin's Theorem (See section 3).

Definition 2.3. Two open sets $P, Q \subseteq M$ are called accessible, if there are points $p \in P, q \in Q$, such that p, q are accessible.

We will call a diffeomorphism f accessible in relation to open sets if any two open sets are accessible.

Obviously accessibility implies accessibility in relation to open sets. The converse is not true in general. It is worth noting that if f is accessible (respectively accessible in relation to open sets) then f^n is accessible (respectively accessible in relation to open sets) for any $n \in \mathbb{Z}$.

Remark 2.1. It is not difficult to prove that accessibility in relation to open sets is equivalent to the existence of a residual set R in M whose points have a dense accessibility class. Indeed if $\{U_k\}_{k \in \mathbb{N}}$ is a countable base of open sets of the manifold and C_k is the set of points accessible to at least one point in U_k , we can take $R = \bigcap_{k \in \mathbb{N}} C_k$ as such a residual set.

Lemma 2.4. Assume that a partially hyperbolic diffeomorphism f has the accessibility property. Then for every $\delta > 0$ there exist $l > 0$ and $R > 0$ such that for any $p, q \in M$ one can find a *us-path* that starts at p , ends within distance $\frac{\delta}{2}$ of q , and has at most l legs, each of them with length at most R .

Proof. See [?]. □

Lemma 2.5. Let $f : M \rightarrow M$ be a partially hyperbolic accessible diffeomorphism. Given $p_0 \in M$, there is $q_0 \in M$ and a *us-path* $z_0(q_0) = p_0, z_1(q_0), \dots, z_N(q_0) = q_0$ connecting p_0 to q_0 and satisfying the following property: for any $\epsilon > 0$ there exist $\delta > 0$ and $L > 0$ such that for every $x \in B(q_0, \delta)$ there exists a *us-path* $z_0(x) = p_0, z_1(x), \dots, z_N(x) = x$ connecting p_0 to x and such that $\text{dist}(z_j(x), z_j(q_0)) < \epsilon$ and $\text{dist}_{\mathcal{F}^\alpha}(z_{j-1}(x), z_j(x)) < L$ for $j = 1, \dots, N$ where $\text{dist}_{\mathcal{F}^\alpha}$ denotes the distance along the strong (either stable or unstable) leaf common to the two points.

Proof. See [?]. □

Lemma 2.6. Assume that a partially hyperbolic diffeomorphism f has the accessibility property. Then there exist $l_0 > 0$ and $R_0 > 0$ such that for any $p, q \in M$ one can find a *us-path* that starts at p , ends at q , and has at most l_0 legs, each of them with length at most R_0 .

Proof. Fix $p_0 \in M$. Let $q_0 \in M$ and a us -path $z_0(q_0) = p_0, z_1(q_0), \dots, z_N(q_0) = q_0$ be as in Lemma 2.5. Let $\epsilon > 0$. Take $\delta > 0$ and $L > 0$ as in Lemma 2.5. For this $\delta > 0$ take $l > 0$ and $R > 0$ as in Lemma 2.4. Next, set $l_0 = 2l + 2N$ and $R_0 = \max\{R, L\}$. Let $p, q \in M$. From Lemma 2.4 we know that there exists a us -path that starts at p (respectively q), ends within distance δ of q_0 , say at p_1 (respectively q_1), and has at most l legs, each of them with length at most R . From Lemma 2.5 there exist a us -path $z_0(p_1) = p_0, z_1(p_1), \dots, z_N(p_1) = p_1$ connecting p_0 to p_1 and a us -path $z_0(q_1) = p_0, z_1(q_1), \dots, z_N(q_1) = q_1$ connecting p_0 to q_1 .

Thus,

$$p_1 = z_N(p_1), z_{N-1}(p_1), \dots, z_0(p_1) = p_0 = z_0(q_1), z_1(q_1), \dots, z_N(q_1) = q_1$$

is a us -path connecting p_1 to q_1 , and it has $2N$ legs, each of them with length at most L . Hence, using the us -path connecting p with p_1 and the us -path connecting q with q_1 , having at most l legs, each of them with length at most R , we have completed the proof. \square

Corollary 2.1. *Let $f : M \rightarrow M$ be a partially hyperbolic accessible diffeomorphism. Then there exist $l_1 > 0$ and $R_1 > 0$ such that for any $p, q \in M$ one can find a us -path $z_0 = p, z_1, \dots, z_{l-1}, z_l = q, l \leq l_1$, that starts at p , ends at q , such that $q \in \mathcal{F}_{R_1}^{ss}(z_{l-1}, f)$, and each leg has length at most R_1 .*

Proof. Let l_0 and R_0 be as in Lemma 2.6. Set $l_1 = l_0 + 1$ and set $R_1 = R_0$. Let $p, q \in M$. Take $q_0 \in \mathcal{F}_{R_0}^{ss}(q, f)$. From Lemma 2.6 we know that one can find a us -path $z_0 = p, z_1, \dots, z_{l-1} = q_0$ that starts at p , ends at q_0 , and has at most l_0 legs, each of them with length at most $R_0 = R_1$. Therefore, $z_0 = p, z_1, \dots, z_{l-1} = q_0, z_l = q$ is a us -path that starts at p , ends at q , and has at most l_1 legs, each of them with length at most R_1 . \square

Now, using the last Corollary, we will prove that if f is accessible then, robustly, for a fixed $r > 0$ and any pair of points $p, q \in M$ there exists a path connecting p to the center unstable disc of radius r centred at q .

Lemma 2.7. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. Given $R_0 > 0$ and $d_0 > 0$ there exist $\delta_0 > 0$ and a neighbourhood $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$ and for every $x, y \in M$ such that $d(x, y) < \delta_0$ we have that $\mathcal{F}_{R_0}^\alpha(x, g)$ and $\mathcal{F}_{R_0}^\alpha(y, f)$ are d_0 -close, $\alpha = ss$ or uu .*

Proof. From Stable Manifold Theorem we know that for every $x \in M$ there exist $r_x > 0$ and a neighbourhood $\mathcal{U}_x(f)$ such that for any $g \in \mathcal{U}_x(f)$ and for every $y \in M$ such that $d(x, y) < r_x$ we have that $\mathcal{F}_{R_0}^\alpha(x, f)$ and $\mathcal{F}_{R_0}^\alpha(y, g)$ are $\frac{d_0}{2}$ -close, $\alpha = ss$ or uu . Thus, for any $g \in \mathcal{U}_x(f)$ and for every $y, z \in B(x, r_x)$ we get $\mathcal{F}_{R_0}^\alpha(y, g)$ and $\mathcal{F}_{R_0}^\alpha(z, f)$ are d_0 -close, $\alpha = ss$ or uu . Since M is compact, there are $x_1, x_2, \dots, x_n \in M$ such that

$$M \subset \bigcup_{i=1}^n B(x_i, r_{x_i}).$$

Let $\delta_0 > 0$ be a Lebesgue number of this cover and take

$$\mathcal{U}(f) = \bigcap_{i=1}^n \mathcal{U}_{x_i}(f).$$

Thus, if $d(x, y) < \delta_0$ then $x, y \in B(x_i, r_{x_i})$ for some $i = 1, 2, \dots, n$. Hence we have that $\mathcal{F}_{R_0}^\alpha(x, f)$ and $\mathcal{F}_{R_0}^\alpha(y, g)$ are d_0 -close, $\alpha = ss$ or uu , for any $g \in \mathcal{U}(f) \subset \mathcal{U}_{x_i}(f)$. \square

Lemma 2.8. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. Given $R_0 > 0$ and $r > 0$ there exist $\epsilon > 0$, $\delta_0 > 0$ and a neighbourhood $\mathcal{V}(f)$ such that for any $g \in \mathcal{V}(f)$ it follows that for any $x, y \in M$ with $d(x, y) < \delta_0$ the following holds:*

$$\mathcal{F}_{R_0+\epsilon}^{ss}(x, g) \cap W_r^{cu}(z, g) \neq \emptyset \quad \text{for any } z \in \mathcal{F}_{R_0}^{ss}(y, f).$$

Proof. Take $\epsilon > 0$ given by Stable Manifold Theorem. There exists a neighbourhood $\mathcal{U}_1(f)$ such that $\epsilon > 0$ can to be taken for any $g \in \mathcal{U}_1(f)$. Let $d_0 > 0$ be such that for any $g \in \mathcal{U}_1(f)$ it follows that if $d(x, y) < d_0$ then

$$\mathcal{F}_\epsilon^{ss}(x, g) \cap W_r^{cu}(y, g) \neq \emptyset.$$

Consider $\mathcal{V}(f) \subset \mathcal{U}_1(f)$ and $\delta_0 > 0$ given by Lemma 2.7. Thus, if $g \in \mathcal{V}(f)$ and $x, y \in M$ with $d(x, y) < \delta_0$ we have that $\mathcal{F}_{R_0}^{ss}(x, g)$ and $\mathcal{F}_{R_0}^{ss}(y, f)$ are d_0 -close and therefore

$$\mathcal{F}_{R_0+\epsilon}^{ss}(x, g) \cap W_r^{cu}(z, g) \neq \emptyset \quad \text{for any } z \in \mathcal{F}_{R_0}^{ss}(y, f).$$

\square

Now we give an easy but interesting consequence of the last two Lemmas, useful for our purposes in Section 4.

Proposition 2.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic accessible diffeomorphism. Given $r > 0$ there exist a neighbourhood $\mathcal{U}(f)$, $l > 0$ and $R > 0$ such that for any $g \in \mathcal{U}(f)$ it follows that for every $p, q \in M$ there exists $q' \in W_r^{cu}(q, g)$ such that one can find a us -path by g that starts at p , ends at q' , and has at most l legs, each of them with length at most R .*

Proof. Let $l_1 > 0$ and $R_1 > 0$ be as in Corollary 2.1. For the sake of simplicity, we will assume that $l_1 = 4$. Given R_1 and r let ϵ , δ_0 and $\mathcal{V}(f)$ be as in Lemma 2.8. From Lemma 2.7 there exist $\delta_1 > 0$ and $\mathcal{U}_1(f) \subset \mathcal{V}(f)$ such that if $g \in \mathcal{U}_1(f)$ and $x, y \in M$ with $d(x, y) < \delta_1$ then $\mathcal{F}_{R_1}^\alpha(x, g)$ and $\mathcal{F}_{R_1}^\alpha(y, f)$ are δ_0 -close, $\alpha = ss$ or uu . Once again, using Lemma 2.7 take $\delta_2 > 0$ and $\mathcal{U}_2(f) \subset \mathcal{U}_1(f)$ such that if $g \in \mathcal{U}_2(f)$ and $x, y \in M$ with $d(x, y) < \delta_2$ then $\mathcal{F}_{R_1}^\alpha(x, g)$ and $\mathcal{F}_{R_1}^\alpha(y, f)$ are δ_1 -close, $\alpha = ss$ or uu . Finally let $\mathcal{U}(f) \subset \mathcal{U}_2(f)$ be a neighbourhood such that for any $g \in \mathcal{U}(f)$ and for any $x \in M$ we have that $\mathcal{F}_{R_1}^\alpha(x, g)$ and $\mathcal{F}_{R_1}^\alpha(x, f)$ are δ_2 -close, $\alpha = ss$ or uu .

Let us prove that $\mathcal{U}(f)$, $l = l_1$ and $R = R_1 + \epsilon$ satisfy what we want. Let $g \in \mathcal{U}(f)$ and let $p, q \in M$. We know that there exists a us -path by f that starts at p , ends at q , and has at most l_1 legs, each of them with length at most R_1 . Moreover, the last leg lies in $\mathcal{F}_{R_1}^{ss}(q, f)$. Suppose that such a us -path has exactly l_1 legs. Let $p = z_0, z_1, z_2, z_3, z_4 = q$ be such a us -path.

We have that $\mathcal{F}_{R_1}^{uu}(p, g)$ and $\mathcal{F}_{R_1}^{uu}(p, f)$ are δ_2 -close. Then, there exists $x_1 \in \mathcal{F}_{R_1}^{uu}(p, g)$ such that $d(x_1, z_1) < \delta_2$. Thus $\mathcal{F}_{R_1}^{ss}(x_1, g)$ and $\mathcal{F}_{R_1}^{ss}(z_1, f)$ are δ_1 -close. Therefore, there exists $x_2 \in \mathcal{F}_{R_1}^{ss}(x_1, g)$ such that $d(x_2, z_2) < \delta_1$. Hence $\mathcal{F}_{R_1}^{uu}(x_2, g)$ and $\mathcal{F}_{R_1}^{uu}(z_2, f)$ are δ_0 -close. Take $x_3 \in \mathcal{F}_{R_1}^{uu}(x_2, g)$ with $d(x_3, z_3) < \delta_0$. From Lemma 2.8, since $q \in \mathcal{F}_{R_1}^{ss}(z_3, f)$ we have that

$$\mathcal{F}_{R_1+\epsilon}^{ss}(x_3, g) \cap W_r^{cu}(q, g) \neq \emptyset.$$

Take

$$q' \in \mathcal{F}_{R_1+\epsilon}^{ss}(x_3, g) \cap W_r^{cu}(q, g),$$

and p, x_1, x_2, x_3, q' the us -path by g . The case that such a us -path by f , connecting p to q , has l' legs with $l' < l_1$, is similar. \square

2.3. Property SH.

We define below the key property introduced in [?] which ensures the robustness of the minimal stable foliation. Moreover, we will prove later that this property also ensures robust transitivity. Before we do, let us introduce some notation: if $L : V \rightarrow W$ is a linear isomorphism between normed vector spaces we denote by $m\{L\}$ the minimum norm of L , i.e. $m\{L\} = \|L^{-1}\|^{-1}$.

Definition 2.4. Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. We say that f exhibits the property SH if there exist $\lambda_0 > 1, C > 0$ such that for any $x \in M$ there exists $y^u(x) \in \mathcal{F}_1^{uu}(x, f)$ (the ball of radius 1 in $\mathcal{F}^{uu}(x, f)$ centred at x) satisfying

$$m\{Df_{|E^c(f^l(y^u(x)))}^n\} > C\lambda_0^n \quad \text{for any } n > 0, \quad l > 0.$$

The Property SH persists under slight perturbations.

Theorem 2.2. Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism exhibiting Property SH. Then, there exist a C^1 neighbourhood \mathcal{U} of f , $C' > 0$ and $\sigma > 1$ such that for any $g \in \mathcal{U}$ it follows that for any $x \in M$ there exists $y^u \in \mathcal{F}_1^{uu}(x, g)$ satisfying

$$m\{Dg_{|E^c(g^l(y^u))}^n\} > C'\sigma^n \quad \text{for any } n > 0, \quad l > 0.$$

Proof. See [?]. \square

Before stating the Theorem which guarantees the robustness of the minimality of a strong stable foliation for a partially hyperbolic diffeomorphism, we recall the concept of minimal stable foliation.

Definition 2.5. Let $f : M \rightarrow M$ be a C^r partially hyperbolic diffeomorphism. We say that $\mathcal{F}^{ss}(f)$ is minimal when $\mathcal{F}^{ss}(x, f)$, the leave of this foliation passing through the point x , is dense in M for every $x \in M$. We say that $\mathcal{F}^{ss}(f)$ is C^r -robustly minimal if there exist a C^r neighbourhood $\mathcal{U}(f)$ such that $\mathcal{F}^{ss}(g)$ is minimal for every diffeomorphism $g \in \mathcal{U}(f)$.

Theorem 2.3. Let $r \geq 1$ and let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism satisfying Property SH and such that the strong stable foliation $\mathcal{F}^{ss}(f)$ is minimal. Then, $\mathcal{F}^{ss}(f)$ is C^1 (and hence C^r) robustly minimal.

Proof. See [?]. \square

2.4. Blenders and Heterodimensional Cycles.

In this subsection we recall the notions of blender and heterodimensional cycle and the relation between them. We also give a condition under which the presence of a blender guarantees the Property SH.

Definition 2.6. A *cs-blender* for $f \in \text{Diff}^r(M)$ with $r \geq 1$ is a hyperbolic set K with a partially hyperbolic structure $E^{ss} \oplus E^c \oplus E^{uu}$ such that $E^c \oplus E^{uu}$ is the unstable bundle, and with a periodic point p such that for any disc D that is C^1 -close to $\mathcal{F}_{loc}^{uu}(p)$, there exists x in the hyperbolic set K such that $\mathcal{F}^{ss}(x)$ intersects D . Moreover, such a property is C^1 -persistent. A *cs-blender* for f^{-1} is called *cu-blender* for f .

Proposition 2.2. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism with strong unstable minimal foliation such that K is a cs-blender for f with a periodic point p . Then f satisfies Property SH.*

Analogously if f has a strong stable minimal foliation and it has a cu-blender then f^{-1} satisfies Property SH.

Proof. It is not difficult to see that if the strong unstable foliation is minimal, then there exists $r > 0$ such that $D_z \subset \mathcal{F}_r^{uu}(z, f), \forall z \in M$, where D_z is a disc C^1 -close to $\mathcal{F}_{loc}^{uu}(p, f)$. Hence, using the Definition 2.6 above, we have that for some $l > 0$ and for every $z \in M$, there exists y^z such that:

$$y^z \in \mathcal{F}_r^{uu}(z, f) \cap \mathcal{F}_l^{ss}(x, f) \quad \text{for some } x \in K.$$

From this, it follows that

$$d(f^n(y^z), f^n(x)) \rightarrow 0 \quad \text{when } n \rightarrow +\infty.$$

Since $T_x M = E^{ss}(x, f) \oplus E^u(x, f) \oplus E^{uu}(x, f)$ for every $x \in K$, $Df|_{E^c(x)}$ is uniformly expanding in the future. Therefore, f satisfies Property SH. \square

Blenders can be produced by unfolding heterodimensional cycles far from homoclinic tangencies as in next Proposition found in [?].

Definition 2.7. Given a diffeomorphism f with two hyperbolic periodic points P_f and Q_f with different indices, say $\text{index}(P_f) > \text{index}(Q_f)$, we say that f has a heterodimensional cycle with codimension $\text{index}(P_f) - \text{index}(Q_f)$ associated to P_f and Q_f if $\mathcal{F}^s(P_f, f)$ and $\mathcal{F}^u(Q_f, f)$ have a (non-trivial) transverse intersection and $\mathcal{F}^u(P_f, f)$ and $\mathcal{F}^s(Q_f, f)$ have a quasi-transverse intersection along the orbit of some point x , i.e., $T_x \mathcal{F}^u(P_f, f) + T_x \mathcal{F}^s(Q_f, f)$ is a direct sum.

Proposition 2.3. *Let f be a C^1 diffeomorphism with a heterodimensional cycle associated to saddles P and Q of indices p and $q = p + 1$. Suppose that the cycle is C^1 -far from homoclinic tangencies. Then there is an open set $\mathcal{V} \subset \text{Diff}^1(M)$ whose closure contains f such that for every g in \mathcal{V} there are a cs-blender defined for g and a cs-blender defined for g^{-1} such that:*

- The cs-blender for g is associated to a hyperbolic periodic point R_g .
- The cs-blender for g^{-1} is associated to a hyperbolic periodic point S_g .

3. PROPERTY SH AND TOPOLOGICALLY MIXING

Our first goal is to show that some diffeomorphisms with Property SH are topologically mixing. In order to do this we will need a few preliminary results.

Lemma 3.1. *Let $\epsilon > 0$ be given by the Stable Manifold Theorem and $r > 0$ sufficiently small. For any $\epsilon' < \epsilon, r' < r$ there exists $d' = d'(\epsilon', r') > 0$ such that for any pair of points $x, y \in M$ with $\text{dist}(x, y) < d'$ the manifolds $W_{r'}^{cu}(x, f)$ and $\mathcal{F}_{\epsilon'}^{ss}(y, f)$ intersect transversally in exactly one point.*

Lemma 3.2. *Given $L > 0$ and $r_0 > 0$ there exist d, r_1 and ϵ_1 with $0 < d < r_1 < r_0$, $\epsilon_1 > 0$ and such that for every $x \in M, z \in W_d^{cu}(x, f)$ if*

$$A_{x,z} = W_{r_1}^{cu}(x, f) \cup \left(\bigcup_{y \in W_d^{cu}(z, f)} \mathcal{F}_{\epsilon_1}^{ss}(y, f) \right)$$

then $\text{diam}(A_{x,z}) < L$ and for any $y \in W_d^{cu}(z, f)$ holds that $\mathcal{F}_{\epsilon_1}^{ss}(y, f)$ intersect $W_{r_1}^{cu}(x, f)$ transversally at exactly one point.

Proof. Let $\epsilon > 0, r > 0$ be as in Lemma 3.1. Take $r_1 < \min\{\frac{L}{8}, r, r_0\}$ and $\epsilon_1 < \min\{\frac{L}{8}, \epsilon\}$. From Lemma 3.1, there exist d_1 such that if $\text{dist}(x, y) < d_1$ then the manifolds $W_{r_1}^{cu}(x, f)$ and $\mathcal{F}_{\epsilon_1}^{ss}(y, f)$ intersect transversally at exactly one point. Choose $d < \min\{\frac{d_1}{4}, r_1\}$. Now let $x \in M$ be arbitrary and $z \in W_d^{cu}(x, f)$. Observe that if $y \in W_d^{cu}(z, f)$ then

$$\text{dist}(x, y) \leq \text{dist}(x, z) + \text{dist}(z, y) \leq \text{dist}_{W_d^{cu}(x, f)}(x, z) + \text{dist}_{W_d^{cu}(z, f)}(z, y) \leq 2d < d_1,$$

so the manifolds $W_{r_1}^{cu}(x, f)$ and $\mathcal{F}_{\epsilon_1}^{ss}(y, f)$ intersect transversally at exactly one point.

Also, for some $y \in W_d^{cu}(z, f)$, if $t \in \mathcal{F}_{\epsilon_1}^{ss}(y, f)$ then

$$\text{dist}(t, x) \leq \text{dist}(t, y) + \text{dist}(y, x) \leq \text{dist}_{\mathcal{F}_{\epsilon_1}^{ss}(y, f)}(t, y) + \text{dist}(y, x) \leq \epsilon_1 + 2d < \frac{L}{2}$$

and if $\bar{t} \in W_{r_1}^{cu}(x, f)$ then

$$\text{dist}(\bar{t}, x) \leq \text{dist}_{W_{r_1}^{cu}(x, f)}(\bar{t}, x) \leq r_1 < \frac{L}{8}.$$

Hence $\text{diam}(A) < L$. □

Proposition 3.1. *If $f : M \rightarrow M$ is a partially hyperbolic diffeomorphism and satisfies Property SH, then for any center-unstable disc D , there exists a periodic hyperbolic point p of stable dimension $\dim(E^{ss})$, whose stable manifold meets D transversally.*

Proof. Let x be a point in M and $D = W_{\beta}^{cu}(x, f)$ a center-unstable disc. For $L = \frac{\beta}{2}$, choose d, r_1, ϵ_1 and $A_{x,z}$ given by Lemma 3.2. Observe that if $t \in W_{r_1}^{cu}(x, f)$ then $d_{W_{r_1}^{cu}(x, f)}(t, x) \leq \text{diam}(A_{x,z}) < L < \beta$. So

$$\mathcal{F}_d^{uu}(x, f) \subset W_d^{cu}(x, f) \subset W_{r_1}^{cu}(x, f) \subset W_{\beta}^{cu}(x, f)$$

Take n_0 such that $\mathcal{F}_1^{uu}(f^{n_0}(x), f) \subset f^{n_0}(\mathcal{F}_d^{uu}(x, f))$. Consider the point $y^u \in \mathcal{F}_1^{uu}(f^{n_0}(x), f)$ satisfying

$$(1) \quad m\{Df_{|E^c(f^l(y^u))}^n\} > C\sigma^n \quad \text{for any } n > 0, \quad l > 0,$$

where $C > 0, \sigma > 1$.

We may assume that $C = 1$. Otherwise we take a fixed power of f . Make $\lambda = \sigma^{-1}$, fix $\lambda_1 \in (\lambda, 1)$ and take r_0 as in Lemma 2.3. Let $\eta > 0$ be such that

$$(2) \quad f^{-n_0}(W_{\eta}^{cu}(y^u, f)) \subset W_d^{cu}(f^{-n_0}(y^u), f).$$

Choose $q \in \omega(y^u)$, ω -limit of y^u , being a recurrent point. For $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(z_1, z_2) < \delta \Rightarrow \mathcal{F}_\epsilon^{ss}(z_1, f) \cap W_{r_0}^{cu}(z_2, f) \neq \emptyset.$$

From the Shadowing Lemma, there exists a periodic hyperbolic point $p \in M$, shadowing a periodic pseudo-orbit in $\omega(y^u)$, constructed by means of the recurrent point q , with $d(p, q) < \frac{\delta}{2}$. Since $q \in \omega(y^u)$, take $m \in \mathbb{N}^*$ such that $\lambda_1^m r_0 < \eta$ and $d(f^m(y^u), q) < \frac{\delta}{2}$. Now set $k_0 = n_0 + m$. Then $d(p, f^m(y^u)) < \delta$ and therefore

$$(3) \quad \mathcal{F}_\epsilon^{ss}(p, f) \cap W_{r_0}^{cu}(f^m(y^u), f) \neq \emptyset.$$

We know that $E^{cu} = E^c \oplus E^u$ is a dominated decomposition. Thus, there is $K > 0$ such that $\|Df|_{E^{cu}}^{-n}\| \leq K \sup\{\|Df|_{E^c}^{-n}\|, \|Df|_{E^u}^{-n}\|\}$. For the sake of simplicity, we will assume that $K = 1$. From (1) we have that

$$\prod_{j=0}^n \|Df|_{E^c(f^{-j}(f^m(y^u)))}^{-1}\| < \lambda^n, \quad 0 \leq n \leq m$$

and therefore

$$\prod_{j=0}^n \|Df|_{E^{cu}(f^{-j}(f^m(y^u)))}^{-1}\| < \lambda^n, \quad 0 \leq n \leq m.$$

From Lemma 2.3 we conclude that $f^{-m}(W_{r_0}^{cu}(f^m(y^u), f)) \subset W_{\lambda_1^m r_0}^{cu}(y^u, f) \subset W_\eta^{cu}(y^u, f)$ and hence, using (2), we have

$$f^{-k_0}(W_{r_0}^{cu}(f^m(y^u), f)) \subset f^{-n_0}(W_\eta^{cu}(y^u, f)) \subset W_d^{cu}(f^{-n_0}(y^u), f).$$

So, from (3),

$$f^{-k_0}(\mathcal{F}_\epsilon^{ss}(p, f)) \cap W_d^{cu}(z, f) \neq \emptyset$$

where $z = f^{-n_0}(y^u) \in \mathcal{F}_d^{uu}(x, f) \subset W_d^{cu}(x, f)$.

Now take $y \in f^{-k_0}(\mathcal{F}_\epsilon^{ss}(p, f)) \cap W_d^{cu}(z, f)$. As $z \in W_d^{cu}(x, f)$, from Lemma 3.2 we conclude that $\mathcal{F}_{\epsilon_1}^{ss}(y, f)$ intersects transversally $W_{r_1}^{cu}(x, f)$ in exactly one point and therefore $\mathcal{F}_\epsilon^{ss}(f^{-k_0}(p))$ intersects transversally $W_{r_1}^{cu}(x, f) \subset D$. \square

Proposition 3.2. *Let $f : M \rightarrow M$ be a partially hyperbolic and transitive diffeomorphism. For any open set U and any center-unstable disc D , there exists a local strong stable disc $\mathcal{F}_\epsilon^{ss}(x, f)$ contained in U with a negative iterated which intersects D transversally.*

Proof. Just take $x \in U$ whose orbit is dense in M and $\epsilon > 0$ such that $\mathcal{F}_\epsilon^{ss}(x, f) \subset U$. There exists a sequence of negative iterates of x , $(f^{n_k}(x))_k$, converging to the center of the disc D . Taking k big enough we can guarantee that $f^{n_k}(\mathcal{F}_\epsilon^{ss}(x, f))$ intersects D transversally. \square

Remark 3.1. Periodic hyperbolic points whose existence is proven in Proposition 3.1 can be taken arbitrarily close to the w -limit of a point z such that $m\{Df|_{E^c(f^l(z))}^n\} > C\lambda_0^n$ for any $n > 0$, $l > 0$ like in Definition 2.4. Consequently these periodic hyperbolic points are chosen uniformly expanding in the central direction.

Theorem 3.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism with the Property SH and such that f^n is transitive for each $n \geq 1$. Then f is topologically mixing.*

Proof. Let $\mathcal{U}, \mathcal{V} \subset M$ be open sets. Take $x \in \mathcal{U}$ arbitrary and $\eta > 0$ such that $D = W_\eta^{cu}(x, f) \subset \mathcal{U}$. From Proposition 3.1, there exists a periodic hyperbolic point p such that $\mathcal{F}^{ss}(p, f)$ intersects D . Assume k to be the period of p . Since f^k is transitive, using Proposition 3.2, there exists a local strong stable disc $\mathcal{F}_\epsilon^{ss}(x, f) \subset \mathcal{W}$ with a negative iterated of f^k , say f^{-kl} , which intersects $W_r^{cu}(p, f)$ for some r sufficiently small. Thus applying λ -Lemma for f^k , we get $n_0 \in \mathbb{N}$ such that

$$\mathcal{F}_\epsilon^{ss}(x, f) \cap f^{nk}(D) \neq \emptyset, \quad \forall n \geq n_0.$$

Hence,

$$f^{-kl}(W) \cap f^{nk}(\mathcal{U}) \neq \emptyset, \quad \forall n \geq n_0.$$

Therefore f^k is topologically mixing. Consequently f is topologically mixing. \square

Now we give our version of Brin's Theorem. Observe that the condition of accessibility in relation to open sets is weaker than the condition of accessibility in the original version of Brin's Theorem.

Let $g \in \text{Diff}^r(M)$. We will denote by $\Omega(g)$ the set of the non-wandering points for g .

Theorem 3.2. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism exhibiting the accessibility property in relation to open sets. If $\Omega(f) = M$ then f is transitive.*

Proof. The proof is similar to the original version in [?]. \square

The condition of f^n being transitive for each $n \geq 1$ is implied by the following Proposition:

Proposition 3.3. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. If $\Omega(f) = M$ then $\Omega(f^n) = M$ for each $n \geq 1$. In particular, $\Omega(f) = M$ and f accessible in relation to open sets imply that f^n is transitive for each $n \geq 1$.*

Proof. Let $n \geq 1$ and let U be an open set in M . As $\Omega(f) = M$ there exist $x_0 \in U$ and $k_0 \geq 1$ such that $f^{k_0}(x_0) \in U$. By continuity, there exists an open set U_0 with $U_0 \subset U$, $x_0 \in U_0$ and $f^{k_0}(U_0) \subset U$. This way we can define recurrently a sequence of points $x_0, x_1, \dots \in M$, a sequence of positive natural numbers k_0, k_1, \dots and a sequence U_0, U_1, \dots of open sets in M such that for any $i \geq 0$ we have $x_{i+1} \in f^{k_i}(U_i)$, $f^{k_{i+1}}(x_{i+1}) \in f^{k_i}(U_i)$, $x_{i+1} \in U_{i+1} \subset f^{k_i}(U_i)$ and $f^{k_{i+1}}(U_{i+1}) \subset f^{k_i}(U_i)$.

It is easy to see that for any pair of non negative integers r, s holds that $f^{-k_r - k_{r+1} - \dots - k_{r+s}}(U_{r+s+1}) \subset U_r$ and $U_r \subset U$. Now define the numbers t_i by $t_i = k_0 + k_1 + \dots + k_i$. Let j and $l \neq 0$ be such that $t_{j+l} \equiv t_j \pmod{n}$. Then there exist an integer m such that $k_{j+1} + \dots + k_{j+l} = mn$. If we set $W = f^{-k_{j+1} - \dots - k_{j+l}}(U_{j+l+1})$ we know that $W \subset U_{j+1} \subset U$ and that $f^{mn}(W) = U_{j+l+1} \subset U$. Hence taking $y \in W \subset U$ we have $f^{mn}(y) \in U$. \square

Remark 3.2. Assume that a partially hyperbolic diffeomorphism f has the accessibility property. If f is transitive then f^n is also transitive for every $n \in \mathbb{Z}^*$.

Corollary 3.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism, accessible in relation to open sets, satisfying $\Omega(f) = M$ and the Property SH, then f is topologically mixing.*

Corollary 3.2. *Let f be a partially hyperbolic diffeomorphism, accessible, topologically transitive and satisfying Property SH. Then f is topologically mixing.*

4. PROPERTY SH AND ACCESSIBILITY

In this section, we follow with other results, which provide facts about the accessibility classes, accessibility in relation to open sets and robust transitivity, considering Property SH.

Theorem 4.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic accessible diffeomorphism exhibiting Property SH. Then there exists a C^1 neighbourhood of f , $\mathcal{U} = \mathcal{U}(f)$, such that for every $g \in \mathcal{U}$ and $p \in M$ it follows that $\mathcal{C}(p, g)$ is dense in M .*

Proof. From Theorem 2.2 we know that there exist a neighbourhood $\mathcal{U}_0(f)$, $C' > 0$ and $\sigma > 1$ such that for every $g \in \mathcal{U}_0$ and $x \in M$ there exists a point $y^u \in \mathcal{F}_1^{uu}(x, g)$ satisfying

$$(4) \quad m\{Dg_{|E^c(g^l(y^u))}^n\} > C'\sigma^n \quad \text{for any } n > 0, \quad l > 0.$$

We may assume that $C = 1$. Otherwise we take a fixed power of every $g \in \mathcal{U}_0$. Let $\lambda = \sigma^{-1}$ and fix $0 < \lambda < \lambda_1 < 1$ and let r be as in Lemma 2.3. For this $r > 0$ take $\mathcal{U}(f) \subset \mathcal{U}_0(f)$, $l > 0$ and $R > 0$ as in Proposition 2.1. We will prove that for every $g \in \mathcal{U}(f)$ and $p \in M$ we have that $\mathcal{C}(p, g)$ is dense in M .

Let $\mathcal{V} \subset M$ be an open set and let $z \in \mathcal{V}$. Let $\beta > 0$ be such that $\mathcal{F}_\beta^{uu}(z, g) \subset \mathcal{V}$. Take n_0 such that $g^{n_0}(\mathcal{F}_\beta^{uu}(z, g)) \supset \mathcal{F}_1^{uu}(g^{n_0}(z), g)$. Consider the point $y^u \in \mathcal{F}_1^{uu}(g^{n_0}(z), g)$ given by Theorem 2.2 and let $\eta > 0$ be such that

$$(5) \quad g^{-n_0}(W_\eta^{cu}(y^u, g)) \subset \mathcal{V}.$$

Choose a positive integer m such that $\lambda_1^m r < \eta$ and set $k = n_0 + m$. From Proposition 2.1 for $q = g^m(y^u)$ there exists $q' \in W_r^{cu}(q, g)$ such that one can find a us -path by g that starts at $g^k(p)$, ends at q' , and has at most l legs, each of them with length at most R .

Since $E^{cu} = E^c \oplus E^u$ and this decomposition is dominated, there is $L > 0$ such that $\|Dg_{|E^{cu}}^{-n}\| \leq L \sup\{\|Dg_{|E^u}^{-n}\|, \|Dg_{|E^c}^{-n}\|\}$. For the sake of simplicity, we will assume that $L = 1$. From (4) we know that

$$\prod_{j=0}^n \|Dg_{|E^c(g^{-j+m}(y^u))}^{-1}\| < \lambda^n, \quad 0 \leq n \leq m$$

and therefore

$$\prod_{j=0}^n \|Dg_{|E^{cu}(g^{-j+m}(y^u))}^{-1}\| < \lambda^n, \quad 0 \leq n \leq m.$$

From Lemma 2.3 we conclude that

$$g^{-m}(W_r^{cu}(g^m(y^u), g)) \subset W_{\lambda_1^m r}^{cu}(y^u, g) \subset W_\eta^{cu}(y^u, g)$$

and hence, using (5), we have $g^{-k}(W_r^{cu}(g^m(y^u), g)) \subset \mathcal{V}$. Since $q' \in W_r^{cu}(g^m(y^u), g)$ we get $g^{-k}(q') \in \mathcal{V}$. Thus, there exists a us -path by g that starts at p , ends at $g^{-k}(q') \in \mathcal{V}$. Hence, $g^{-k}(q') \in \mathcal{V} \cap \mathcal{C}(p, g)$ and the proof is completed. \square

Corollary 4.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic accessible diffeomorphism exhibiting Property SH. Then there exists a C^1 neighbourhood of f , $\mathcal{U} = \mathcal{U}(f)$, such that for any $g \in \mathcal{U}$ it follows that g is accessible in relation to open sets.*

Corollary 4.2. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic accessible diffeomorphism exhibiting Property SH and such that $\Omega(f) = M$. Then any diffeomorphism g being C^1 -close to f and such that $\Omega(g) = M$ is topologically mixing.*

Corollary 4.3. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic, accessible, volume preserving diffeomorphism exhibiting the Property SH, then any diffeomorphism C^1 -close to f that is volume preserving is topologically mixing.*

5. PROPERTY SH AND ROBUST TRANSITIVITY

Unlike the results in preceding section our next Theorem do not have in the hypotheses the condition of Accessibility. Property SH is enough to guarantee robust transitivity.

Lemma 5.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. There exist $\epsilon > 0$ such that given $r > 0$ there are $\delta > 0$ and a neighbourhood \mathcal{V}_0 of f such that for any $x, y \in M$ with $d(x, y) < \delta$ it follows that*

- $\mathcal{F}_\epsilon^{ss}(x, g) \cap \mathcal{W}_r^{cu}(y, g) \neq \emptyset$
- $\mathcal{F}_\epsilon^{uu}(x, g) \cap \mathcal{W}_r^{cs}(y, g) \neq \emptyset$, for any $g \in \mathcal{V}_0$.

Proof. The result follows from Stable Manifold Theorem. \square

Theorem 5.1. *Let M be a compact Riemannian manifold and let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism, non-hyperbolic, transitive. If f and f^{-1} satisfy Property SH then f is robustly transitive.*

Proof. For the sake of clarity we divide the proof in two steps. The first step deals with the construction of an appropriate neighborhood \mathcal{V} of f . In the second step we prove that any diffeomorphism in \mathcal{V} is transitive.

Step 1

From Theorem 2.2 there exist a neighborhood $\mathcal{V}_1(f)$, $C_0 > 0$ and $\sigma_0 > 1$ such that for every $g \in \mathcal{V}_1$ and $x \in M$ there exists a point $y \in \mathcal{F}_1^{uu}(x, g)$ such that

$$(6) \quad m\{Dg_{|E^c(g^l(y))}^n\} > C_0 \sigma_0^n \quad \text{for any } n > 0, \quad l > 0.$$

Analogously there exist a neighborhood $\mathcal{V}_2(f^{-1})$, $C_1 > 0$ and $\sigma_1 > 1$ such that for every $h \in \mathcal{V}_2$ and $x \in M$ there exists a point $y \in \mathcal{F}_1^{uu}(x, h)$ such that

$$m\{Dh_{|E^c(h^l(y))}^n\} > C_1 \sigma_1^n \quad \text{for any } n > 0, \quad l > 0.$$

Take $C = \min\{C_0, C_1\} > 0$ and $\sigma = \min\{\sigma_0, \sigma_1\} > 1$. Thus, for every $g \in \mathcal{V}_1 \cup \mathcal{V}_2$ and $x \in M$ there exists a point $y \in \mathcal{F}_1^{uu}(x, g)$ such that

$$m\{Dg_{|E^c(g^l(y))}^n\} > C \sigma^n \quad \text{for any } n > 0, \quad l > 0.$$

We may assume that $C = 1$. Otherwise we take a fixed power of every $g \in \mathcal{V}_1 \cup \mathcal{V}_2$. Let $\mathcal{V}_3(f) \subset \mathcal{V}_1$ be a neighborhood of f such that if $g \in \mathcal{V}_3$ then $g^{-1} \in \mathcal{V}_2$. Let $\lambda = \sigma^{-1}$, fix $0 < \lambda < \lambda_1 < 1$ and let $r > 0$ be as in Lemma 2.3. Consider $\epsilon > 0$ given by Stable Manifold Theorem and let $r > 0$ be as above. Take $\delta > 0$ and take $\mathcal{V}_4(f) \subset \mathcal{V}_3$ a neighborhood of f as in Lemma 5.1. Since f is transitive there

exists a point $z \in M$ such that $\{f^n(z); n \in \mathbb{N}\}$ and $\{f^{-n}(z); n \in \mathbb{N}\}$ are dense in M . Therefore

$$M = \bigcup_{n \in \mathbb{N}} B(f^n(z), \frac{\delta}{2})$$

and by compactness there exist positive integers $n_1 < \dots < n_l$ such that

$$\bigcup_{i=1}^l B(f^{n_i}(z), \frac{\delta}{2}) = M.$$

Next, choose a positive integer m_0 and a neighborhood $\mathcal{V}_5(f) \subset \mathcal{V}_4$ such that if $m \geq m_0$, $g \in \mathcal{V}_5$ and $q \in M$ then

- $g^m(\mathcal{F}_\epsilon^{ss}(q, g)) \subset B(g^m(q), \frac{\delta}{6})$
- $g^{-m}(\mathcal{F}_\epsilon^{uu}(q, g)) \subset B(g^{-m}(q), \frac{\delta}{6})$

Affirmation 1. For each $i = 2, \dots, l$ there exists $m_i \in \mathbb{Z}_+^*$ satisfying:

$$(i) \quad f^{m_i}(z) \in B(f^{n_i}(z), \frac{\delta}{6}) \quad \text{for } i = 2, \dots, l$$

$$(ii) \quad \begin{aligned} m_2 &> n_1 + m_0 \\ m_i &> m_{i-1} + m_0 \quad \text{for } i = 3, \dots, l \end{aligned}$$

Proof. It follows by density of $\{f^n(z); n \in \mathbb{N}\}$ in M . □

Affirmation 2. For each $i = 2, \dots, l$ there exists $\overline{m}_i \in \mathbb{Z}_-^*$ satisfying:

$$(iii) \quad f^{\overline{m}_i}(z) \in B(f^{n_i}(z), \frac{\delta}{6}) \quad \text{for } i = 2, \dots, l$$

$$(iv) \quad \begin{aligned} \overline{m}_2 &< n_1 - m_0 \\ \overline{m}_i &< \overline{m}_{i-1} - m_0 \quad \text{for } i = 3, \dots, l \end{aligned}$$

Proof. It follows by density of $\{f^{-n}(z); n \in \mathbb{N}\}$ in M . □

Set $l_0 = \max\{n_l, m_2, m_3, m_4, \dots, m_l, -\overline{m}_2, -\overline{m}_3, \dots, -\overline{m}_l\}$.

Observe that $l_0 \geq n_l > n_i$ for $i = 1, \dots, l-1$.

Take a neighborhood $\mathcal{V}(f) \subset \mathcal{V}_5$ such that $d_{C^0}(g^n, f^n) < \frac{\delta}{6}$, for any $n \in \mathbb{Z}$ with $|n| \leq l_0$, for any $g \in \mathcal{V}$.

Step 2

We will prove that any $g \in \mathcal{V}$ is transitive. Take two arbitrary open sets $\mathcal{U}, \mathcal{W} \subset M$. Let us prove that there exists a positive integer k_0 such that $g^{k_0}(\mathcal{U}) \cap \mathcal{W} \neq \emptyset$. Let $u \in \mathcal{U}$ and $w \in \mathcal{W}$. Let $\beta > 0$ be such that $\mathcal{F}_\beta^{uu}(u, g) \subset \mathcal{U}$ and $\mathcal{F}_\beta^{uu}(w, g^{-1}) \subset \mathcal{W}$. Take n_0 such that $g^{n_0}(\mathcal{F}_{\frac{\beta}{2}}^{uu}(u, g)) \supset \mathcal{F}_1^{uu}(g^{n_0}(u), g)$ and $g^{-n_0}(\mathcal{F}_{\frac{\beta}{2}}^{uu}(w, g^{-1})) \supset \mathcal{F}_1^{uu}(g^{-n_0}(w), g^{-1})$. Consider $y \in \mathcal{F}_1^{uu}(g^{n_0}(u), g)$ and $x \in \mathcal{F}_1^{uu}(g^{-n_0}(w), g^{-1})$ satisfying:

$$(7) \quad \begin{cases} m\{Dg_{|E^c}^n(g^l(y))\} > \sigma^n & \text{for any } n > 0, \quad l > 0 \\ m\{Dg_{|E^c}^{-n}(g^{-l}(x))\} > \sigma^n & \text{for any } n > 0, \quad l > 0. \end{cases}$$

Observe that

- $\mathcal{F}_{\frac{\beta}{2}}^{uu}(g^{-n_0}(y), g) \subset \mathcal{U}$ because $g^{-n_0}(y) \in \mathcal{F}_{\frac{\beta}{2}}^{uu}(u, g) \subset \mathcal{F}_{\beta}^{uu}(u, g) \subset \mathcal{U}$
- $\mathcal{F}_{\frac{\beta}{2}}^{uu}(g^{n_0}(x), g^{-1}) \subset \mathcal{W}$ because $g^{n_0}(x) \in \mathcal{F}_{\frac{\beta}{2}}^{uu}(w, g^{-1}) \subset \mathcal{F}_{\beta}^{uu}(w, g^{-1}) \subset \mathcal{W}$.

Thus, there exist $A \subset \mathcal{U}$ a neighborhood of $g^{-n_0}(y)$ and $B \subset \mathcal{W}$ a neighborhood of $g^{n_0}(x)$ such that

$$(8) \quad \begin{cases} \mathcal{F}_{\frac{\beta}{2}}^{uu}(a, g) \subset \mathcal{U} & \text{for any } a \in A \\ \mathcal{F}_{\frac{\beta}{2}}^{uu}(b, g^{-1}) \subset \mathcal{W} & \text{for any } b \in B. \end{cases}$$

Let $\eta > 0$ be such that

$$(9) \quad \begin{cases} g^{-n_0}(\mathcal{W}_{\eta}^{cu}(y, g)) \subset A \subset \mathcal{U} \\ g^{n_0}(\mathcal{W}_{\eta}^{cu}(x, g^{-1})) \subset B \subset \mathcal{W} \end{cases}$$

Next, choose a positive integer m' such that $\lambda_1^{m'} r < \eta$ and

$$(10) \quad \begin{cases} g^{m'+n_0}(\mathcal{F}_{\frac{\beta}{2}}^{uu}(q, g)) \supset \mathcal{F}_{\epsilon}^{uu}(g^{m'+n_0}(q), g) & \text{for any } q \in M \\ g^{-(m'+n_0)}(\mathcal{F}_{\frac{\beta}{2}}^{uu}(q, g^{-1})) \supset \mathcal{F}_{\epsilon}^{uu}(g^{-(m'+n_0)}(q), g^{-1}) & \text{for any } q \in M \end{cases}$$

Set $k' = n_0 + m'$. Thus, using (16), we get

$$\begin{aligned} \prod_{j=0}^{n-1} \|Dg_{|_{E^c(g^{-j}(y))}}^{-1}\| &< \lambda^n, & 0 \leq n \leq m' \\ \prod_{j=0}^{n-1} \|Dg_{|_{E^c(g^j(x))}}\| &< \lambda^n, & 0 \leq n \leq m' \end{aligned}$$

and therefore

$$\begin{aligned} \prod_{j=0}^{n-1} \|Dg_{|_{E^{cu}(g^{-j}(y))}}^{-1}\| &< \lambda^n, & 0 \leq n \leq m' \\ \prod_{j=0}^{n-1} \|Dg_{|_{E^{cu}(g^j(x))}}\| &< \lambda^n, & 0 \leq n \leq m' \end{aligned}$$

From Lemma 2.3 we conclude that

$$(11) \quad \begin{cases} g^{-m'}(\mathcal{W}_r^{cu}(g^{m'}(y), g)) \subset \mathcal{W}_{\lambda_1^{m'} r}^{cu}(y, g) \subset \mathcal{W}_{\eta}^{cu}(y, g) \\ g^{m'}(\mathcal{W}_r^{cu}(g^{-m'}(x), g^{-1})) \subset \mathcal{W}_{\lambda_1^{m'} r}^{cu}(x, g^{-1}) \subset \mathcal{W}_{\eta}^{cu}(x, g^{-1}) \end{cases}$$

and hence, using (17), we have

$$(12) \quad \begin{cases} g^{-k'}(\mathcal{W}_r^{cu}(g^{m'}(y), g)) \subset A \subset \mathcal{U} \\ g^{k'}(\mathcal{W}_r^{cu}(g^{-m'}(x), g^{-1})) \subset B \subset \mathcal{W}. \end{cases}$$

Particularly, it follows

$$(13) \quad \begin{cases} \mathcal{W}_r^{cu}(g^{m'}(y), g) \subset g^{k'}(\mathcal{U}) \\ \mathcal{W}_r^{cu}(g^{-m'}(x), g^{-1}) \subset g^{-k'}(\mathcal{W}). \end{cases}$$

Moreover, if $p \in \mathcal{W}_r^{cu}(g^{m'}(y), g)$, $q \in \mathcal{W}_r^{cu}(g^{-m'}(x), g^{-1})$ then $g^{-k'}(p) \in A$ and $g^{k'}(q) \in B$ due to (21). Thus, from (8),

$$\mathcal{F}_{\frac{\beta}{2}}^{uu}(g^{-k'}(p), g) \subset \mathcal{U} \quad \text{and} \quad \mathcal{F}_{\frac{\beta}{2}}^{uu}(g^{k'}(q), g^{-1}) \subset \mathcal{W}$$

and hence, (10) imply

$$(14) \quad \begin{cases} \mathcal{F}_{\epsilon}^{uu}(p, g) \subset g^{k'}(\mathcal{F}_{\frac{\beta}{2}}^{uu}(g^{-k'}(p), g)) \subset g^{k'}(\mathcal{U}) \\ \mathcal{F}_{\epsilon}^{uu}(q, g^{-1}) \subset g^{-k'}(\mathcal{F}_{\frac{\beta}{2}}^{uu}(g^{k'}(q), g^{-1})) \subset g^{-k'}(\mathcal{W}). \end{cases}$$

Finally, from (22) and (14) we conclude that

$$\begin{aligned} (i) \quad & \mathcal{W}_r^{cu}(g^{m'}(y), g) \subset g^{k'}(\mathcal{U}) \\ (ii) \quad & \mathcal{F}_{\epsilon}^{uu}(p, g) \subset g^{k'}(\mathcal{U}), \quad \forall p \in \mathcal{W}_r^{cu}(g^{m'}(y), g) \\ (iii) \quad & \mathcal{W}_r^{cs}(g^{-m'}(x), g) \subset g^{-k'}(\mathcal{W}) \\ (iv) \quad & \mathcal{F}_{\epsilon}^{ss}(q, g) \subset g^{-k'}(\mathcal{W}), \quad \forall q \in \mathcal{W}_r^{cs}(g^{-m'}(x), g) \end{aligned}$$

For the sake of simplicity, we will denote $g^{m'}(y)$ for \tilde{y} and $g^{-m'}(x)$ for \tilde{x} .

Since $M = \bigcup_{i=1}^l B(f^{n_i}(z), \frac{\delta}{2})$, there are $i, j \in \{1, \dots, l\}$ such that

$$\tilde{y} \in B(f^{n_i}(z), \frac{\delta}{2}) \quad \text{and} \quad \tilde{x} \in B(f^{n_j}(z), \frac{\delta}{2}).$$

- Case $i = j$

In this case, $d(\tilde{x}, \tilde{y}) < \delta$. Thus, using Lemma 5.1, $\mathcal{F}_{\epsilon}^{uu}(\tilde{y}, g) \cap W_r^{cs}(\tilde{x}, g) \neq \emptyset$. Moreover, by (ii) and by (iii), we have that $\mathcal{F}_{\epsilon}^{uu}(\tilde{y}, g) \subset g^{k'}(\mathcal{U})$ and $\mathcal{W}_r^{cs}(\tilde{x}, g) \subset g^{-k'}(\mathcal{W})$, and hence, $g^{k'}(\mathcal{U}) \cap g^{-k'}(\mathcal{W}) \neq \emptyset$, i.e., $g^{2k'}(\mathcal{U}) \cap \mathcal{W} \neq \emptyset$.

Next, we will prove the case $i < j$. The case $i > j$ is similar.

- Case $i < j$

First assume $i > 1$. Consider $j = i + k$ for $k = 1, 2, \dots, l - i$. In this case we have that

$$\begin{aligned} d(\tilde{y}, g^{m_i}(z)) &\leq d(\tilde{y}, f^{n_i}(z)) + d(f^{n_i}(z), f^{m_i}(z)) + d(f^{m_i}(z), g^{m_i}(z)) \\ &< \frac{\delta}{2} + \frac{\delta}{6} + \frac{\delta}{6} < \delta \end{aligned}$$

and therefore

$$\mathcal{F}_{\epsilon}^{ss}(g^{m_i}(z), g) \cap W_r^{cu}(\tilde{y}, g) \neq \emptyset.$$

Take $p \in \mathcal{F}_{\epsilon}^{ss}(g^{m_i}(z), g) \cap W_r^{cu}(\tilde{y}, g)$. Since

$$\begin{aligned}
m_j - m_i &= m_{i+k} - m_i = (m_{i+k} - m_{i+(k-1)}) + (m_{i+(k-1)} - m_{i+(k-2)}) \\
&\quad + \cdots + (m_{i+1} - m_i) > km_0 > m_0, \\
g^{m_j-m_i}(\mathcal{F}_\epsilon^{ss}(g^{m_i}(z), g)) &\subset B(g^{m_j}(z), \frac{\delta}{6}),
\end{aligned}$$

and from this it follows that

$$g^{m_j-m_i}(p) \in B(g^{m_j}(z), \frac{\delta}{6}).$$

Thus,

$$\begin{aligned}
d(g^{m_j-m_i}(p), \tilde{x}) &\leq d(g^{m_j-m_i}(p), g^{m_j}(z)) + d(g^{m_j}(z), f^{m_j}(z)) \\
&\quad + d(f^{m_j}(z), f^{n_j}(z)) + d(f^{n_j}(z), \tilde{x}) \\
&< \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{2} = \delta
\end{aligned}$$

and from Lemma 5.1, we get

$$(v) \quad \mathcal{F}_\epsilon^{uu}(g^{m_j-m_i}(p), g) \cap \mathcal{W}_r^{cs}(\tilde{x}, g) \neq \emptyset.$$

Using that $(m_j - m_i) > 0$ and $p \in \mathcal{W}_r^{cu}(\tilde{y}, g)$ and using (ii), we have that

$$\mathcal{F}_\epsilon^{uu}(g^{m_j-m_i}(p), g) \subset g^{m_j-m_i}(\mathcal{F}_\epsilon^{uu}(p, g)) \subset g^{m_j-m_i}(g^{k'}(\mathcal{U})).$$

From (iii) and (v) we conclude that

$$g^{m_j-m_i}(g^{k'}(\mathcal{U})) \cap g^{-k'}(\mathcal{W}) \neq \emptyset.$$

In this case the proof is completed.

Now, assume $i = 1$.

Consider $j = i + k$ for $k = 1, 2, \dots, l - i$. In this case we have that

$$\begin{aligned}
d(\tilde{y}, g^{n_1}(z)) &\leq d(\tilde{y}, f^{n_1}(z)) + d(f^{n_1}(z), g^{n_1}(z)) \\
&< \frac{\delta}{2} + \frac{\delta}{6} < \delta
\end{aligned}$$

and therefore

$$\mathcal{F}_\epsilon^{ss}(g^{n_1}(z), g) \cap \mathcal{W}_r^{cu}(\tilde{y}, g) \neq \emptyset.$$

Take $p \in \mathcal{F}_\epsilon^{ss}(g^{n_1}(z), g) \cap \mathcal{W}_r^{cu}(\tilde{y}, g)$. Since

$$\begin{aligned}
m_j - n_1 &= m_{1+k} - n_1 = (m_{1+k} - m_k) + (m_k - m_{k-1}) + \cdots + (m_2 - n_1) \\
&> km_0 > m_0, \\
g^{m_j-n_1}(\mathcal{F}_\epsilon^{ss}(g^{n_1}(z), g)) &\subset B(g^{m_j}(z), \frac{\delta}{6}),
\end{aligned}$$

and from this it follows that

$$g^{m_j-n_1}(p) \in B(g^{m_j}(z), \frac{\delta}{6}).$$

Thus,

$$\begin{aligned} d(g^{m_j-n_1}(p), \tilde{x}) &\leq d(g^{m_j-n_1}(p), g^{m_j}(z)) + d(g^{m_j}(z), f^{m_j}(z)) \\ &\quad + d(f^{m_j}(z), f^{n_j}(z)) + d(f^{n_j}(z), \tilde{x}) \\ &< \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta}{2} = \delta \end{aligned}$$

and from Lemma 5.1, we get

$$(vi) \quad \mathcal{F}_\epsilon^{uu}(g^{m_j-n_1}(p), g) \cap \mathcal{W}_r^{cs}(\tilde{x}, g) \neq \emptyset.$$

However,

$$\mathcal{F}_\epsilon^{uu}(g^{m_j-n_1}(p), g) \subset g^{m_j-n_1}(\mathcal{F}_\epsilon^{uu}(p, g)) \subset g^{m_j-n_1}(g^{k'}(\mathcal{U}))$$

due to

$$m_j - n_1 > 0, \quad p \in \mathcal{W}_r^{cu}(\tilde{y}, g) \quad \text{and} \quad (ii).$$

From (iii) and (vi) we conclude that

$$g^{m_j-n_1}(g^{k'}(\mathcal{U})) \cap g^{-k'}(\mathcal{W}) \neq \emptyset.$$

Hence, the case $i < j$ is completed. The case $i > j$ follows by symmetry, and the proof of Theorem is completed. \square

The proof of last Theorem suggests the following Proposition as a possible, future, alternative way to remove the condition Property SH for f^{-1} , to get robust transitivity.

Proposition 5.1. *Let M be a compact Riemannian manifold, f a partially hyperbolic C^r -diffeomorphism in M , and p a periodic hyperbolic point for f , whose central direction is unstable and with a $\frac{\delta}{4}$ -dense orbit, δ as in Lemma 5.1. If f satisfy Property SH then f is robustly transitive.*

Proof. Analogously to the proof in Theorem 5.1 we divide the proof in two steps. The first step deals with the construction of an appropriate neighbourhood \mathcal{V} of f . In the second step we prove that any diffeomorphism in \mathcal{V} is transitive.

Step 1

From Theorem 2.2 there exist a C^1 -neighbourhood, $\mathcal{V}_1(f)$, $C > 0$ and $\sigma > 1$ such that for every $g \in \mathcal{V}_1$ and $x \in M$ there exists a point $y \in \mathcal{F}_1^{uu}(x, g)$ such that

$$(15) \quad m\{Dg_{|E^c(g^l(y))}^n\} > C\sigma^n \quad \text{for any} \quad n > 0, \quad l > 0.$$

We may assume that $C = 1$. Otherwise we take a fixed power of every $g \in \mathcal{V}_1$. Let $\lambda = \sigma^{-1}$, fix $0 < \lambda < \lambda_1 < 1$ and let $r > 0$ be as in Lemma 2.3. Consider $\epsilon > 0$ given by Stable Manifold Theorem and let $r > 0$ be as above. Take $\delta > 0$ and $\mathcal{V}_2(f) \subset \mathcal{V}_1$ a neighbourhood of f as in Lemma 5.1. Next choose a neighbourhood $\mathcal{V}(f)$ contained in \mathcal{V}_2 such that if $g \in \mathcal{V}$ then the hyperbolic continuation p_g of p is a hyperbolic periodic point of g with unstable central direction and a $\frac{\delta}{2}$ -dense orbit.

Step 2

We will prove that any $g \in \mathcal{V}$ is transitive. Take two arbitrary open sets $\mathcal{U}, \mathcal{W} \subset M$. Let us prove that there exists a positive integer k_0 such that $g^{k_0}(\mathcal{U}) \cap \mathcal{W} \neq \emptyset$. Choose $u \in \mathcal{U}$ and $x \in \mathcal{W}$. Let $\beta > 0$ be such that $\mathcal{F}_\beta^{uu}(u, g) \subset \mathcal{U}$ and $\mathcal{F}_\beta^{ss}(x, g) \subset \mathcal{W}$. Take n_0 such that $g^{n_0}(\mathcal{F}_{\frac{\beta}{2}}^{uu}(u, g)) \supset \mathcal{F}_1^{uu}(g^{n_0}(u), g)$. Consider $y \in \mathcal{F}_1^{uu}(g^{n_0}(u), g)$ satisfying:

$$(16) \quad m\{Dg_{|E^c}^n(g^l(y))\} > \sigma^n \quad \text{for any } n > 0, \quad l > 0$$

Let $\eta > 0$ be such that

$$(17) \quad g^{-n_0}(\mathcal{W}_\eta^{cu}(y, g)) \subset \mathcal{U}$$

Next, choose a positive integer m' such that $\lambda_1^{m'} r < \eta$. Set $k' = n_0 + m'$. Thus, using (16), we get

$$(18) \quad \prod_{j=0}^{n-1} \|Dg_{|E^c(g^{-j}(y))}^{-1}\| < \lambda^n, \quad 0 \leq n \leq m'$$

and therefore

$$(19) \quad \prod_{j=0}^{n-1} \|Dg_{|E^{cu}(g^{-j}(y))}^{-1}\| < \lambda^n, \quad 0 \leq n \leq m'$$

From Lemma 2.3 we conclude that

$$(20) \quad g^{-m'}(\mathcal{W}_r^{cu}(g^{m'}(y), g)) \subset \mathcal{W}_{\lambda_1^{m'} r}^{cu}(y, g) \subset \mathcal{W}_\eta^{cu}(y, g)$$

and hence, using (17), we have

$$(21) \quad g^{-k'}(\mathcal{W}_r^{cu}(g^{m'}(y), g)) \subset \mathcal{U}$$

Particularly, it follows

$$(22) \quad \mathcal{W}_r^{cu}(g^{m'}(y), g) \subset g^{k'}(\mathcal{U})$$

For the sake of simplicity, we will denote $g^{m'}(y)$ for \tilde{y} and $g^{-t}(x)$ for \tilde{x} . Now choose a positive integer t such that

$$(23) \quad \mathcal{F}_\epsilon^{ss}(g^{-t}(w), g) \subset g^{-t}(\mathcal{F}_\beta^{ss}(w, g))$$

Let L be the period of p_g . There exist $i, j \in \{0, 1, \dots, L-1\}$ such that $\tilde{y} \in B(g^i(p_g), \frac{\delta}{2})$ and $\tilde{x} \in B(g^j(p_g), \frac{\delta}{2})$.

- Case $i = j$

In this case, $d(\tilde{x}, \tilde{y}) < \delta$. Thus, using Lemma 5.1, $\mathcal{F}_\epsilon^{ss}(\tilde{x}, g) \pitchfork W_r^{cu}(\tilde{y}, g) \neq \emptyset$. Moreover, we have $\mathcal{F}_\epsilon^{ss}(\tilde{x}, g) \subset g^{-t}(W)$ and $W_r^{cu}(\tilde{y}, g) \subset g^{k'}(\mathcal{U})$ so $g^{-t}(W) \cap g^{k'}(\mathcal{U}) \neq \emptyset$ that is $\mathcal{W} \cap g^{k'+t}(\mathcal{U}) \neq \emptyset$.

Next, we will prove the case $i \neq j$.

- Case $i \neq j$

We may assume without loss of generality that $i < j$. Consider $j = i + k$ for some $k \in \{1, 2, \dots, L - i\}$. Hence $d(\bar{y}, g^i(p_g)) < \frac{\delta}{2}$ and $\mathcal{F}_\epsilon^{ss}(g^i(p_g), g) \cap W_r^{cu}(\bar{y}, g) \neq \emptyset$. Take $q \in \mathcal{F}_\epsilon^{ss}(g^i(p_g), g) \cap W_r^{cu}(\bar{y}, g)$. Note that $g^{j-i}(W_r^{cu}(\bar{y}, g))$ intersect transversally $\mathcal{F}_\epsilon^{ss}(g^i(p_g), g)$ in $g^{j-i}(q)$. Using the Lambda-Lemma for g^L we get the existence of $N_0 \in \mathbb{N}$ such that $g^{nL}(g^{j-i}(D))$ is C^1 -close to $W_r^{cu}((g^i(p_g)), g)$ for any $n \geq N_0$ where D is a disc such that $q \in D \subset W_r^{cu}(\bar{y}, g)$.

Also as $\bar{x} \in B(g^i(p_g), \frac{\delta}{2})$ then $\mathcal{F}_\epsilon^{ss}(\bar{x}, g) \cap W_r^{cu}(g^j(p_g), g) \neq \emptyset$. So there exists $N_1 \in \mathbb{N}$ with $N_1 > N_0$ such that $g^{nL}(g^{j-i}(D)) \cap \mathcal{F}_\epsilon^{ss}(\bar{x}, g) \neq \emptyset$ for any $n \geq N_1$. So $g^{nL}(g^{j-i}(g^{k'}(\mathcal{U}))) \cap g^{-t}(\mathcal{W}) \neq \emptyset$ for any $n \geq N_1$. Consequently g is transitive. \square

6. PROPERTY SH AND DENSITY OF PERIODIC POINTS

Here we prove that for a diffeomorphism exhibiting Property SH and minimality of the strong stable foliation the set of its periodic points is dense. So both transitivity and density of the periodic points are robust properties under the hypotheses of Property SH and minimality of the strong stable foliation.

Theorem 6.1. *Let $f \in \text{Diff}^r(M)$ be a partially hyperbolic diffeomorphism exhibiting Property SH and such that the strong stable foliation is minimal. Then, $\overline{\text{Per}(f)} = M$.*

Proof.

Remark 6.1. Changing f by a power of itself, we can assume that there is $\sigma > 1$ such that for any $x \in M$ there exists $y^u \in \mathcal{F}_1^{uu}(x, f)$ such that

$$(24) \quad m\{Df_{|E^c(f^l(y^u))}^n\} > \sigma^n \text{ for any } n > 0, \quad l > 0.$$

Let SH be defined by:

$$(25) \quad SH = \{y \in M : m\{Df_{|E^c(f^l(y))}^n\} > \sigma^n \text{ for any } n > 0, \quad l > 0\}.$$

Lemma 6.1. *If $SH \subset \overline{\text{Per}(f)}$ then $M \subset \overline{\text{Per}(f)}$.*

Proof. Given $x \in M$ and V an open set containing x choose $\beta > 0$ be such that $\mathcal{F}_\beta^{uu}(x, f) \subset V$ and l_0 such that $\mathcal{F}_1^{uu}(f^{l_0}(x), f) \subset f^{l_0}(\mathcal{F}_\beta^{uu}(x, f))$. Then take $h \in \mathcal{F}_1^{uu}(f^{l_0}(x), f) \cap SH$ and use continuity of f . \square

From now on our goal will be to prove that $SH \subset \overline{\text{Per}(f)}$.

Let us fix ϵ', r' and d' as in Lemma 3.1.

Definition 6.1. We will call a cylinder any open set $W \subset M$, with $\text{diam}(W) < d'$, which is the domain of some local chart $\eta : M \rightarrow \mathbb{R}^n$ trivializing the strong stable foliation such that $W_{r'}^{cu}(y, f) \not\subset W$ and $\mathcal{F}_{\epsilon'}^{ss}(y, f) \not\subset W$ for any $y \in W$.

Lemma 6.2. *For every $x \in M$ there exists a cylinder containing x .*

Proof. First observe that there exists a local chart $(\widetilde{W}, \widetilde{\eta})$, trivializing the strong stable foliation, with $x \in \widetilde{W}$ and such that $W_{r'}^{cu}(x, f) \not\subset \widetilde{W}$, $\mathcal{F}_{\epsilon'}^{ss}(x, f) \not\subset \widetilde{W}$. Now by the continuous dependence of the manifolds $W_{r'}^{cu}(y, f)$ and $\mathcal{F}_{\epsilon'}^{ss}(y, f)$ on the point y , follows the existence of an open set $\widetilde{\widetilde{W}} \subset \widetilde{W}$ containing x and such that $W_{r'}^{cu}(z, f) \not\subset \widetilde{\widetilde{W}}$, $\mathcal{F}_{\epsilon'}^{ss}(z, f) \not\subset \widetilde{\widetilde{W}}$ for any $z \in \widetilde{\widetilde{W}}$. Finally take a local chart trivializing the strong stable foliation (W, η) , with $x \in W \subset \widetilde{\widetilde{W}}$ and $\text{diam}(W) < d'$. \square

Notice that there exists a base B of open sets of M whose elements are cylinders. Let \mathcal{C} be an open covering of cylinders of the manifold M and L its Lebesgue number.

Lemma 6.3. *Let C be a cylinder and let $\eta : C \rightarrow U^{cu} \times V^{ss}$ be a local chart trivializing the strong stable foliation, where U^{cu} , V^{ss} are open sets in \mathbb{R}^{c+uu} , \mathbb{R}^{ss} respectively and $0 \in \eta(C)$. Let $\pi : U^{cu} \times V^{ss} \rightarrow U^{cu} \times \{0\}$ be the projection of $\mathbb{R}^{c+uu+ss}$ on $\mathbb{R}^{c+uu} \times \{0\}$. Let $h \in C$ and $\hat{r} > 0$ be such that $\overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)} \subset C$ and let $\hat{\delta} > 0$ be such that $\overline{W_{\hat{\delta}}^{cu}(y, f)} \subset C$ for any $y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$. Denote $\eta(h)$ by (h_{cu}, h_{ss}) . Then the following hold:*

- (i) $\pi|_{\eta(\overline{W_{\hat{\delta}}^{cu}(y, f)})}$ is an homeomorphism on its image for any $y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$.
- (ii) There exists an open ball $B \subset U^{cu}$ centered at h_{cu} such that

$$B \times \{0\} \subset \bigcap_{y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}} \pi(\eta(W_{\hat{\delta}}^{cu}(y, f))).$$

- (iii) There exists $0 < \bar{\delta} < \hat{\delta}$ such that for any $y_1, y_2 \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$ there exists a continuous map $\pi_{y_1, y_2} : W_{\bar{\delta}}^{cu}(y_1, f) \rightarrow W_{\bar{\delta}}^{cu}(y_2, f)$ and if $t' = \pi_{y_1, y_2}(t)$ then $t' \in \mathcal{F}^{ss}(t, f)$.

Proof. (i) Observe that as C is a cylinder if $\overline{W_{\hat{\delta}}^{cu}(y, f)} \cap \mathcal{F}_{\hat{r}}^{ss}(x, f) \neq \emptyset$ then this intersection is exactly one point and $\pi|_{\eta(\overline{W_{\hat{\delta}}^{cu}(y, f)})}$ is injective.

(ii) As $\pi|_{\eta(\overline{W_{\hat{\delta}}^{cu}(y, f)})}$ is an homeomorphism on its image by the Invariance of Domain Theorem for any $y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$ there exists an open ball $B_y \subset U^{cu}$ centered at h_{cu} such that $B_y \times \{0\} \subset \pi(\eta(W_{\hat{\delta}}^{cu}(y, f)))$. Now use the compacity of $\overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$ and continuous dependence of the manifolds $W_{\hat{\delta}}^{cu}(y, f)$ on the points y .

(iii) Choose $\bar{\delta} < \hat{\delta}$ such that $\eta(W_{\bar{\delta}}^{cu}(y, f)) \subset B \times V^{ss}$ for any $y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$. Then define:

$$\pi_{y_1, y_2} = (\eta)^{-1} \circ (\pi|_{\eta(\overline{W_{\bar{\delta}}^{cu}(y_2, f)})})^{-1} \circ \pi|_{\eta(\overline{W_{\bar{\delta}}^{cu}(y_1, f)})} \circ \eta : W_{\bar{\delta}}^{cu}(y_1, f) \rightarrow W_{\bar{\delta}}^{cu}(y_2, f)$$

□

Choose $\delta > 0$ such that if $\text{dist}(z, SH) < \delta$ then

$$(26) \quad \|Df|_{E^c(f(z))}^{-1}\| < (\sigma')^{-1} < 1$$

for some $1 < \sigma' < \sigma$.

Let us define the set SH' by

$$SH' = \bigcup_{z \in SH} \mathcal{F}_{\delta}^{ss}(z, f).$$

Lemma 6.4. *If $x \in SH'$ then $m\{Df|_{E^c(f^l(x))}^n\} > (\sigma')^n$ for any $n > 0, l > 0$.*

Proof. It follows by induction, using (26) and the fact that $f(SH) \subset SH$. □

Let $\alpha = (\sigma')^{-1}$, fix α_1 with $0 < \alpha < \alpha_1 < 1$ and let r_0 be as in Lemma 2.3.

Consider $\lambda < 1$ the contraction factor of the strong stable subbundle.

Let $h \in SH$ and let $U \subset M$ be an open set containing h . We will prove that there exists a periodic point in U .

Let $C \in B$ be a cylinder contained in U such that $h \in C$.

Take \hat{r}, K and $n_0 \in \mathbb{N}$ such that $0 < \hat{r} < \delta$, $\overline{\mathcal{F}_{2\hat{r}}^{ss}(h, f)} \subset C$, $K > \delta + \epsilon_1$,

$$(27) \quad \mathcal{F}_K^{ss}(x, f) \cap W_d^{cu}(y, f) \neq \emptyset, \forall x, y \in M.$$

and

$$\mathcal{F}_K^{ss}(f^{-n_0}(x), f) \subset f^{-n_0}(\mathcal{F}_{\hat{r}}^{ss}(x, f)), \quad \forall x \in M.$$

Take also $\hat{\delta}$ satisfying simultaneously the following three conditions:

- 1) $\overline{W_{\hat{\delta}}^{cu}(y, f)} \subset C$, $\forall y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$
- 2) $f^{-n_0}(W_{\hat{\delta}}^{cu}(y, f)) \subset W_d^{cu}(f^{-n_0}(y), f)$, $\forall y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$
- 3) If $\text{dist}(z, \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}) < \hat{\delta}$ then $\overline{\mathcal{F}_{\hat{r}}^{ss}(z, f)} \subset C$.

Let now $\bar{\delta}$ and π_{y_1, y_2} be like in Lemma 6.3 and such that

$$(28) \quad \mathcal{F}_{\hat{r}}^{ss}(\pi_{h, y_2}(q), f) \subset \mathcal{F}_{\bar{\delta}}^{ss}(q, f), \quad \forall q \in W_{\bar{\delta}}^{cu}(h, f), \quad \forall y_2 \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$$

Let $N \in \mathbb{N}$ be such that $(\alpha_1)^N r_0 < \bar{\delta}$ and $f^N(\overline{\mathcal{F}_{2\hat{r}}^{ss}(y, f)}) \subset \mathcal{F}_{\bar{\delta}}^{ss}(f^N(y), f)$, $\forall y \in M$. From Lemma 6.4 it follows that

$$\prod_{j=0}^n \|Df_{|E^c(f^{-j}(z))}^{-1}\| < \alpha^n, \quad 0 \leq n \leq N, \quad \forall z \in f^N(\overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)})$$

and therefore

$$\prod_{j=0}^n \|Df_{|E^{cu}(f^{-j}(z))}^{-1}\| < \alpha^n, \quad 0 \leq n \leq N, \quad \forall z \in f^N(\overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}).$$

Then by Lemma 2.3 we conclude that:

$$f^{-N}(W_{r_1}^{cu}(f^N(y), f)) \subset f^{-N}(W_{r_0}^{cu}(f^N(y), f)) \subset W_{\alpha_1^N r_0}^{cu}(y, f) \subset W_{\bar{\delta}}^{cu}(y, f)$$

for any $y \in \overline{\mathcal{F}_{\hat{r}}^{ss}(h, f)}$.

Put $y_1 = h$. We know that $\mathcal{F}_K^{ss}(f^{-n_0}(h), f)$ intersects $W_d^{cu}(f^N(h), f)$ in some point z . Then there exists $y_2 \in \mathcal{F}_{\hat{r}}^{ss}(h, f)$ such that $z = f^{-n_0}(y_2)$ and a continuous function $\pi_{h, y_2} : W_{\bar{\delta}}^{cu}(h, f) \rightarrow W_{\bar{\delta}}^{cu}(y_2, f)$ such that if $t' = \pi_{h, y_2}(t)$ then $t' \in \mathcal{F}^{ss}(t, f)$.

Set $x = f^N(h)$. Lemma 3.2 implies the existence of a cylinder \hat{C} containing

$$A = W_{r_1}^{cu}(x, f) \cup \left(\bigcup_{y \in W_d^{cu}(z, f)} \mathcal{F}_{\epsilon_1}^{ss}(y, f) \right)$$

and such that for any $y \in W_d^{cu}(z, f)$ the intersection of the manifolds $\mathcal{F}_{\epsilon_1}^{ss}(y, f)$ and $W_{r_1}^{cu}(x, f)$ is exactly one point.

Now let $\phi : \hat{C} \rightarrow \hat{U}^{cu} \times \hat{V}^{ss}$ be the trivializing local chart of the strong stable foliation with $0 \in \phi(\hat{C})$ and $\hat{\pi} : \hat{U}^{cu} \times \hat{V}^{ss} \rightarrow \hat{U}^{cu} \times \{0\}$ the projection. Observe that if $\pi_1 = \hat{\pi}|_{\phi(\overline{W_{r_1}^{cu}(x, f)})}$ then π_1 is a homeomorphism on its image.

On the other side if $\pi_2 = \hat{\pi}|_{\phi(W_d^{cu}(z, f))}$, as the intersection between the manifolds $\mathcal{F}_{\epsilon_1}^{ss}(y, f)$ and $W_{r_1}^{cu}(x, f)$ is exactly one point, it follows

$$\pi_2(\phi(W_d^{cu}(z, f))) \subset \pi_1(\phi(W_{r_1}^{cu}(x, f))).$$

Then the function

$$g : \pi_1(\phi(\overline{W_{r_1}^{cu}(x, f)})) \rightarrow \pi_2(\phi(W_d^{cu}(z, f)))$$

defined by $g = \pi_2 \circ \phi \circ f^{-n_0} \circ \pi_{h,y_2} \circ f^{-N} \circ (\phi)^{-1} \circ (\pi_1)^{-1}$ is well defined and continuous. Hence by Brower's fixed point Theorem there exists a fixed point $p \in \pi_1(\phi(W_{r_1}^{cu}(x, f)))$, that is

$$(29) \quad \pi_2 \circ \phi \circ f^{-n_0} \circ \pi_{h,y_2} \circ f^{-N} \circ (\phi)^{-1} \circ (\pi_1)^{-1}(p) = p.$$

Observe that $\hat{\pi}(\pi_1^{-1}(p)) = \hat{\pi}(\pi_2^{-1}(p)) = p$ so if $p_{r_1} = (\phi)^{-1} \circ \pi_1^{-1}(p)$, $p_{-n_0} = (\phi)^{-1} \circ \pi_2^{-1}(p)$ then $\mathcal{F}^{ss}(p_{r_1}, f) = \mathcal{F}^{ss}(p_{-n_0}, f)$ and

$$\mathcal{F}^{ss}(f^{-N}(p_{-n_0}), f) = \mathcal{F}^{ss}(f^{-N}(p_{r_1}), f) \text{ and } \mathcal{F}^{ss}(f^{n_0}(p_{-n_0}), f) = \mathcal{F}^{ss}(f^{n_0}(p_{r_1}), f).$$

From (29)

$$\pi_{h,y_2} \circ f^{-N}(p_{r_1}) = f^{n_0}(p_{-n_0}).$$

So

$$\begin{aligned} \mathcal{F}^{ss}(f^{n_0}(p_{r_1}), f) &= \mathcal{F}^{ss}(f^{n_0}(p_{-n_0}), f) = \mathcal{F}^{ss}(\pi_{h,y_2} \circ f^{-N}(p_{r_1}), f) \\ &= \mathcal{F}^{ss}(f^{-N}(p_{r_1}), f) = \mathcal{F}^{ss}(f^{-N}(p_{-n_0}), f) \end{aligned}$$

which implies $\mathcal{F}^{ss}(p_{r_1}, f) = \mathcal{F}^{ss}(f^{-n_0-N}(p_{-n_0}), f)$.

Observe that $p_{-n_0} \in W_d^{cu}(z, f)$, $p_{r_1} \in W_{r_1}^{cu}(x, f) = W_{r_1}^{cu}(f^N(h), f)$ and $p_{r_1} \in \mathcal{F}_{\epsilon_1}^{ss}(p_{-n_0}, f)$. Thus $\text{dist}_{\mathcal{F}^{ss}}(p_{r_1}, p_{-n_0}) \leq \epsilon_1$ and $p_{-n_0} = f^{-n_0} \circ \pi_{h,y_2} \circ f^{-N}(p_{r_1}) \in f^{-n_0}(W_{\hat{\delta}}^{cu}(y_2, f))$.

Take $\theta \in \mathcal{F}_{\delta}^{ss}(p_{r_1}, f)$ arbitrary. Then

$$\text{dist}_{\mathcal{F}^{ss}}(\theta, p_{-n_0}) \leq \text{dist}_{\mathcal{F}^{ss}}(\theta, p_{r_1}) + \text{dist}_{\mathcal{F}^{ss}}(p_{r_1}, p_{-n_0}) \leq \delta + \epsilon_1 < K$$

and from there

$$\mathcal{F}_{\delta}^{ss}(p_{r_1}, f) \subset \mathcal{F}_K^{ss}(p_{-n_0}, f) \subset f^{-n_0}(\mathcal{F}_{\hat{\delta}}^{ss}(f^{n_0}(p_{-n_0}), f)).$$

Remember that

$$\begin{aligned} f^N(\overline{\mathcal{F}_{\hat{\delta}}^{ss}(f^{n_0}(p_{-n_0}), f)}) &\subset f^N(\overline{\mathcal{F}_{2\hat{\delta}}^{ss}(f^{-N}(p_{r_1}), f)}) \subset \mathcal{F}_{\delta}^{ss}(f^N(f^{-N}(p_{r_1})), f) \\ &= \mathcal{F}_{\delta}^{ss}(p_{r_1}, f) \subset f^{-n_0}(\mathcal{F}_{\hat{\delta}}^{ss}(f^{n_0}(p_{-n_0}), f)) \end{aligned}$$

From there

$$\begin{aligned} f^{n_0+N}(\overline{f^{-n_0}(\mathcal{F}_{\hat{\delta}}^{ss}(f^{n_0}(p_{-n_0}), f))}) &= f^N(\overline{\mathcal{F}_{\hat{\delta}}^{ss}(f^{n_0}(p_{-n_0}), f)}) \\ &\subset f^{-n_0}(\overline{\mathcal{F}_{\hat{\delta}}^{ss}(f^{n_0}(p_{-n_0}), f)}). \end{aligned}$$

Again by Brower's fixed point Theorem there exists $Q \in f^{-n_0}(\overline{\mathcal{F}_{\hat{\delta}}^{ss}(f^{n_0}(p_{-n_0}), f)}) \subset f^{-n_0}(C)$ a fixed point by the function f^{n_0+N} , and hence $f^{n_0}(Q)$ a periodic point in C .

□

7. EXAMPLES

7.1. Shub's example.

The conditions in Corollary 3.1 and Theorem 6.1 are fulfilled by the widely known example of Shub. This is because Pujals-Sambarino proved in [?] that it satisfies the Property SH and that its stable foliation is robustly minimal.

7.2. A wider scenario for SH on the inverse.

The following Proposition 7.1 show how, under some conditions, to get perturbations with the Property SH and whose inverses also has the Property SH. All the conditions in this Proposition are fulfilled, in particular, by Mañé's example.

Let M be a smooth compact boundaryless three dimensional manifold and \mathcal{T} the set of non Anosov robustly transitive partially hyperbolic diffeomorphisms in M . Denote by \mathcal{T}' the subset of \mathcal{T} consisting of the diffeomorphisms with strong stable robustly minimal foliation.

Proposition 7.1. *There exists an open and dense subset \mathcal{D}' of \mathcal{T}' such that for every $g \in \mathcal{D}'$ we have that g^{-1} satisfies Property SH.*

Proof. Just apply the following Claim 4 and Proposition 2.2. □

Claim 1. *Let $f \in \text{Diff}^1(M)$ be a diffeomorphism such that its periodic points are C^1 -robustly hyperbolic and $\Omega(f) = M$. Then f is Anosov.*

Proof. See [?]. □

Claim 2. *There is a dense subset \mathcal{A} of \mathcal{T} such that for every $f \in \mathcal{A}$ there exists a pair of hyperbolic periodic points with different indices.*

Proof. It follows from Claim 1 and [?]. □

Claim 3. *There exists a dense subset \mathcal{B} of \mathcal{T} such that every $f \in \mathcal{B}$ has a heterodimensional cycle of codimension one.*

Proof. It follows from Claim 2 and [?]. □

Claim 4. *There exists an open and dense subset \mathcal{D} of \mathcal{T} such that every $g \in \mathcal{D}$ has a cs-blender and a cu-blender.*

Proof. It follows from Claim 3 and Proposition 2.3. □